

# THE PARAMETERS SPACE OF THE SPIN-ORBIT PROBLEM I. NORMALLY HYPERBOLIC INVARIANT CIRCLES

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**ABSTRACT.** In this paper we start a global study of the parameter space (dissipation, perturbation, frequency) of the dissipative spin-orbit problem in Celestial Mechanics with the aim of delimiting regions where the dynamics, or at least some of its important features, is determined. This is done via a study of the corresponding family of time  $2\pi$ -maps. In the same spirit as Chenciner in [6], we are at first interested in delimiting regions where the normal hyperbolicity is sufficiently important to guarantee the persistence of an invariant attractive (resp. repulsive) circle under perturbation. As a tool, we use an analogue for diffeomorphisms in this family of Rüssmann's translated curve theorem in analytic category.

## 1. INTRODUCTION

The starting point of this paper is the persistence of Diophantine quasi-periodic attractive tori in the dissipative spin-orbit model given in [23]; this problem, first investigated by Celletti and Chierchia in [5], is presented as follows.

**The spin-orbit problem.** This problem concerns the rotations of a tri-axial non rigid satellite (or planet), whose center of mass revolves on a given Keplerian elliptic orbit, focused on a fixed massive point. The equation of motion is derived making the following assumptions:

- the body is modeled as an ellipsoid
- the spin axis of the body coincides with the smallest physical axis (the major axis of inertia) and is perpendicular to the orbital plane (hence obliquity of the spin-axis is ignored)
- the internal structure of the body is non rigid and non-elastic
- the satellite is in (or very close to) spin-orbit resonance with its planet and is supposed to be in slow rotating regime<sup>1</sup>

Under these hypothesis, the rotation's dynamics is described by the following equation in  $\mathbb{R}$

$$(1.1) \quad \ddot{\theta} + \eta(\dot{\theta} - \nu) + \varepsilon \partial_{\theta} f(\theta, t) = 0.$$

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<sup>1</sup>A satellite (or planet) is said to be in  $n : k$  spin-orbit resonance when it rotates  $n$  times about its spin-axis while revolving exactly  $k$  times around its planet (or star).

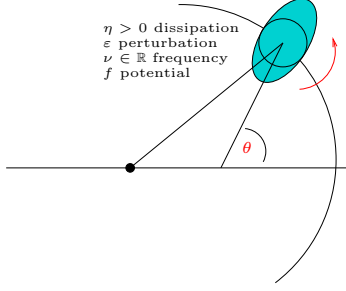


FIGURE 1. The spin-orbit problem

In the equation

- $\theta, t$  are  $2\pi$ -periodic variables  $(\theta, t) \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ , and  $\dot{\theta} \in \mathbb{R}$
- $\varepsilon = \frac{B-A}{C} > 0$  ( $A, B, C$  being the principal moments of inertia of the body) measures the size of the perturbation, indeed the oblateness of the satellite: when calculating the potential exercised by the satellite, if the two equatorial axes are not of the same length, in addition to the Keplerian potential there is the so called "tidal potential", containing a coupling between  $\theta$  and the distance between the center of mass of the satellite and the planet. Since here the Keplerian orbit is given, the potential function is indicated just as  $f(\theta, t)$  ( $f$  real analytic in  $(\theta, t) \in \mathbb{T}^2$ ).
- $\nu \in \mathbb{R}$  is an external free parameter representing the proper frequency of the attractor of the dynamics when  $\varepsilon = 0$  (see below to see how it intervenes in the dynamics).
- $\eta\dot{\theta}$ , with  $\eta \in \mathbb{R}^+$ , is a friction term due to the non rigidity of the internal structure and its response to gravitational forces.

A realistic approximation of the satellite's deformation with the tidal frequency is given by the "visco-elastic model" (see [12, 13]) and its eventual simplifications due to particular values of tidal frequencies. In the model of interest to us, the position of "maximal tide" of the satellite is supposed to be shifted from the attractive-point by a constant time lag; this assumption together with the hypothesis of slow rotating regime and the spin-orbit resonance, translates in the presence of the linear dissipative interaction term  $\eta(\dot{\theta} - \nu)$ .

The complete derivation of (1.1) would take several pages; since it is not the main focus of this paper, we send the interested reader to [4, 10–13, 15] and references therein for a complete physical discussion of the model and deduction of the equation.

The main question is then the existence of KAM quasi-periodic solutions of (1.1) when  $\eta \neq 0$ . In their approach Celletti-Chierchia look for a function  $u : \mathbb{T}^2 \rightarrow \mathbb{R}$  such that the solution of (1.1) can be written as

$$\theta(t) = \alpha t + u(\alpha t, t),$$

$\alpha$  being a fixed Diophantine frequency.

Provided that  $\varepsilon$  is small enough, for any  $\eta \in [-\eta_0, \eta_0], \eta_0 \in \mathbb{R}^+$  the function  $u$  is eventually found as the solution of an opportune PDE, for a particular value of  $\nu$  (see [5, Theorem 1] for the precise statement). In [31] Stefanelli-Locatelli give a generalization of this result in higher dimension extending to the dissipative case the classical 1954 proof of Kolmogorov on the existence of quasi-periodic motions in Hamiltonian systems; while in [23, 24] the persistence of quasi-periodic attractive tori is proved at the help of suitable normal forms results in the spirit of Moser, Herman and Rüssmann [19, 25, 27], and the technique of "elimination of parameters" (see [3, 6–8, 14, 19, 29] at instance). Since the persistence result given in the spirit of [23, 24] is the starting point of the investigation carried out in the present paper, we will recall it here.

**Curves of quasi-periodic tori.** In [23] the spin-orbit result is deduced in the following frame.

Let  $\mathcal{V}^{\text{Ham}} \oplus (-\eta r + \zeta) \partial_r$  be the space of (germs of) real analytic Hamiltonian vector fields with non degenerate torsion on  $\mathbb{T}^n \times \mathbb{R}^n$  extended with the non-Hamiltonian term  $-\eta r + \zeta$ ,  $\zeta \in \mathbb{R}^n$  in the  $r$ -direction<sup>2</sup> and  $\mathcal{U}^{\text{Ham}}(\alpha, -\eta)$  be its affine subspace of vector fields of the form

$$u(\theta, r) = (\alpha + O(r), -\eta r + O(r^2)),$$

where  $\alpha \in \mathbb{R}^n, \eta \in \mathbb{R}$  are fixed and  $O(r^k)$  stands for terms of order  $\geq k$  in  $r$  eventually depending on  $\theta$ , vanishing when  $r = 0$ . In particular,  $\alpha$  is supposed to satisfy the following Diophantine condition for some real  $\gamma, \tau > 0$

$$|k \cdot \alpha| \geq \frac{\gamma}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}.$$

Let  $\mathcal{G}^\omega$  be the space of germs of real analytic symplectic transformations

$$g(\theta, r) = (\varphi(\theta), {}^t\varphi'^{-1}(\theta) \cdot (r + \rho(\theta))),$$

where  $\varphi$  is a diffeomorphism of the torus fixing the origin and  $\rho$  be a closed 1-form on the torus.

The "translated torus" theorem [23, Theorem 6.1] asserts that if  $v$  is close enough to some  $u^0 \in \mathcal{U}^{\text{Ham}}(\alpha, -\eta)$ , there exists unique  $(g, u, b) \in \mathcal{G}^\omega \times \mathcal{U}^{\text{Ham}}(\alpha, -\eta) \times \mathbb{R}^n$ , in the neighborhood of  $(\text{id}, u^0, 0)$  such that

$$(1.2) \quad g_* u + b \partial_r = v,$$

where  $g_* u = (g' \cdot u) \circ g^{-1}$  indicates the push-forward of  $u$  by  $g$ . The image of the torus  $g(\mathbb{T}^n)$  is translated by  $b$  by the flow of  $v$ .

The spin-orbit problem is eventually handled with this normal form.

In fact, by introducing a frequency  $\alpha \in \mathbb{R}$  satisfying the Diophantine condition

$$(1.3) \quad |k\alpha - l| \geq \frac{\gamma}{|k|^\tau} \quad \forall (k, l) \in \mathbb{N} \setminus \{0\} \times \mathbb{Z},$$

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<sup>2</sup>We noted  $r = (r_1, \dots, r_n)$  and omitted the tensor product sign  $r \otimes \partial_r$ .

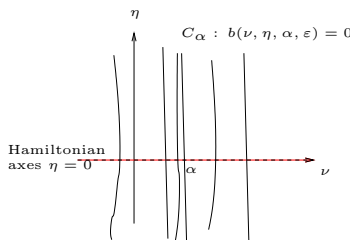
for some  $\gamma, \tau > 0$ , we see that the vector field corresponding to (1.1) in the coordinates  $(\theta, r = \dot{\theta} - \alpha)$  reads

$$(1.4) \quad v = \begin{cases} \dot{\theta} = \alpha + r \\ \dot{r} = -\eta r + \eta(\nu - \alpha) - \varepsilon \partial_{\theta} f(\theta, t). \end{cases}$$

Evidently, when  $\varepsilon = 0$ ,  $T_0 = \mathbb{T} \times \{r = 0\}$  is an invariant attractive quasi-periodic torus for  $v$  provided that  $\nu = \alpha$ .

Adapting the previous frame to this non-autonomous system, in [23, corollary 7.1] we proved that (1.4) admits a normal form like (1.2), in the neighborhood of  $(\text{id}, u^0 = (\alpha + r, -\eta r), \eta(\nu - \alpha))$ , and that there exists a unique "frequency adjustment"  $\nu$  that solves  $b = 0$ , thus proving that the spin-orbit system possesses an invariant normally-attractive quasi-periodic torus (see [23, Theorem 7.1]). The normal form being analytic with respect to the perturbation  $\varepsilon$  and smooth with respect to the parameters  $(\nu, \eta)$ , we gave the portrait entailed by the following corollary.<sup>3</sup>

**Corollary 1.1.** *Fixing  $\alpha$  Diophantine and  $\varepsilon$  sufficiently small, there exists a unique curve  $C_{\alpha}$  (analytic in  $\varepsilon$ , smooth in  $\eta$ ) in the plane  $(\eta, \nu)$  of the form  $\nu = \alpha + O(\varepsilon^2)$ , along which the translation  $b = b(\nu, \alpha, \eta, \varepsilon)$  vanishes, so that the perturbed system (1.4) possesses an invariant torus carrying quasi-periodic motion of frequency  $\alpha$ . This torus is normally attractive (resp. repulsive) if  $\eta > 0$  (resp.  $\eta < 0$ ).*



The aim of this work is to understand what happens for values of parameters  $(\eta, \nu)$  in the complement of the Cantor set of curves  $C_{\alpha}$ .

**The parameters space: zones of "normal hyperbolicity".** In 1985 A. Chenciner started an analysis of the dynamical properties of generic 2-parameter families of germs of diffeomorphism of  $\mathbb{R}^2$  which unfold an elliptic fixed point. In [6–8], he showed that along a certain curve  $\Gamma$  in the space of parameters, we find all the complexity that the dynamics of a germ of generic area preserving diffeomorphism of  $\mathbb{R}^2$  presents, in the neighborhood of an elliptic fixed point.

In the same spirit of A. Chenciner in [6], the present paper starts from results recalled above with the purpose of "thickening" the curves  $C_{\alpha}$  and showing that in some well chosen regions containing them, the persistence of invariant attractive

<sup>3</sup>These results are part of the Ph.D thesis [24] of the present author; they are referenced in it as corollary 3.2.1, theorem 3.2 and corollary 3.2.2. The work is available at <http://dynamical-systems.eu/wp-content/uploads/2015/11/Tesi.pdf>

(resp. repulsive) tori is again guaranteed thanks to their "normal hyperbolicity".

It is well known that invariant manifolds of dimension greater than 0 systematically persist only if they are "normally hyperbolic" (see [20] and [22]). Roughly speaking, if  $P$  is a  $C^1$ -diffeomorphism of a smooth manifold  $M$  and  $V \subseteq M$  a submanifold invariant by  $P$ ,  $V$  is normally hyperbolic if the tangent map  $TP$  restricted to the normal direction to  $V$  dominates the restriction of  $TP$  to the tangent direction  $TV$ .

A precise definition will be given in section 2.1, in the case of interest to us.<sup>4</sup> Hence, if the normal attraction is strong enough, Hadamard's "graph transform" method (see [16]) allows to find the invariant perturbed manifold as the fixed point of a contraction in a well chosen functional space.

The heart of the present work is the power of suitable changes of variables and successive localizations that allow to delimit regions in the space of parameters where the graph transform method apply. Let us be more precise.

This study starts from the general solution of (1.4) when  $\varepsilon = 0$

$$\begin{cases} \theta(t) = \theta(0) + \nu t + [r(0) - (\nu - \alpha)] \frac{1 - e^{-\eta t}}{\eta}, \\ r(t) = r(0) + (e^{-\eta t} - 1)[r(0) - (\nu - \alpha)]. \end{cases}$$

The period of the perturbation being  $2\pi$ , we are interested in the map

$$(1.5) \quad P(\theta(0), r(0)) = (\theta(2\pi), r(2\pi)).$$

The circle  $r = r(0)$  is "translated" by the quantity

$$\tau = r(2\pi) - r(0) = (e^{-2\pi\eta} - 1)(r(0) - (\nu - \alpha))$$

and "rotated" by the angle

$$\begin{aligned} \theta(2\pi) - \theta(0) &= 2\pi\nu + [r(0) - (\nu - \alpha)] \frac{1 - e^{-2\pi\eta}}{\eta} \\ &= 2\pi\nu - \frac{\tau}{\eta}. \end{aligned}$$

In particular, the unique circle which is rotated by an angle  $2\pi\alpha$  is the one with radius

$$r_\alpha = (\nu - \alpha) \left[ 1 + \frac{2\pi\eta}{e^{-2\pi\eta} - 1} \right];$$

this circle is translated by the quantity

$$\tau_\alpha = 2\pi\eta(\nu - \alpha).$$

The center of our interest will be the map

$$\begin{cases} P(\theta, r) = (\theta', r') \\ \theta' = \theta + 2\pi\nu + \frac{(1 - e^{-2\pi\eta})}{\eta}(r - (\nu - \alpha)) \\ r' = (\nu - \alpha) + e^{-2\pi\eta}(r - (\nu - \alpha)) \end{cases}$$

and its real analytic perturbations.

At first, we localize our study in a neighborhood of the invariant circle of rotation number  $2\pi\nu$  and prove that for high enough values of the dissipation  $\eta$ , this circle

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<sup>4</sup>This property is extensively used in bifurcation theory (Hopf bifurcation, Denjoy attractors for example; see [9, 17, 26] and references therein at instance).

persists under the perturbation, no matter what  $2\pi\nu$  is. It results a first region where the normal hyperbolicity prevails (see theorem 2.1).

Then, we will adapt to our context Rüssmann's translated curve theorem [18, 27] in order to perform a second localization (see section 3), and use all the strength of the Diophantine properties of  $\alpha$  to put the perturbation  $Q$  of  $P$  in a meaningful normal form from which we will identify a new bigger region in which the normal hyperbolicity is strong enough to imply the existence of an invariant normally hyperbolic circle (see section 3, theorem 3.1).

In both situations the method allows to deduce that the basins of attraction of the invariant circles coincide with the whole phase space, thus providing a global control on the dynamics, for values of parameters of which the normal hyperbolicity prevails.

A figure summarizing the improvements and the regions where invariant attractive circles persist will conclude this study.

## 2. INVARIANT CIRCLES FOR ANY ROTATION NUMBER

Corollary 1.1 guarantees the existence, in a plane  $\varepsilon = \text{const.}$  in the space  $(\eta, \nu, \varepsilon)$ , of a Cantor set of curves

$$C_\alpha := b(\nu(\eta, \alpha, \varepsilon) = 0$$

along which the invariant attractive torus with Diophantine frequency  $\alpha$  persists under perturbation. We recall that all along  $C_\alpha$  the perturbed vector field  $v$  can be written in the form

$$v = (\alpha + O(\tilde{r}), -\eta\tilde{r} + O(\tilde{r}^2)),$$

showing that  $\eta = 0$  is the only value of transition between the attractive and the repulsive regime of the invariant torus.

**2.1. The strength of dissipation: graph transform.** Let  $V$  be a smooth compact submanifold of a smooth manifold  $M$  and suppose that  $P : M \rightarrow M$  is a  $C^r$  diffeomorphism and  $P(V) = V$ .

We say that  $P$  is  $r$ -normally hyperbolic at  $V$  (or that  $V$  is  $r$ -normally hyperbolic) if the tangent bundle of  $M$  over  $V$  has a  $TP$ -invariant splitting

$$(2.1) \quad TM|_V = TV \oplus N^s \oplus N^u,$$

and for all  $x \in V$ ,

$$(2.2) \quad \begin{cases} \sup_x \|TP|_{N_x^s}\| < \inf_x m(TP|_{T_x V})^k, \\ \inf_x m(TP|_{N_x^u}) > \sup_x \|TP|_{T_x V}\|^k \end{cases} \quad 1 \leq k \leq r.$$

We recall that the minimum norm  $m(A)$  of a linear transformation  $A$  is defined as  $m(A) = \inf_{|y|=1} |Ay|$ .

Condition (2.2) expresses domination of the normal behavior over the tangent behavior of  $TP$ .<sup>5</sup>

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<sup>5</sup>There are many equivalent definitions of normal hyperbolicity (see [1, 20, 30]); we used the strongest given by Hirsh-Pugh-Shub in [20].

In the case of our interest, this definition becomes extremely straight. In fact, the real analytic diffeomorphism  $P$  of  $\mathbb{T} \times \mathbb{R}$  defined by (1.5), after the coordinate's change  $(\theta, r) \mapsto (\theta, r - (\nu - \alpha) = \rho)$  reads

$$P(\theta, \rho) = (\theta + 2\pi\nu + \frac{1 - e^{-2\pi\eta}}{\eta}\rho, \rho e^{-2\pi\eta}).$$

The circle  $C_0 = \mathbb{T} \times \{\rho = 0\}$  is clearly  $r$ -normally hyperbolic; the tangent bundle being trivial the given definition hides no ambiguity.

In particular the invariant splitting (2.1) is reduced to the sum of the tangent bundle  $TC_0$  and a normal attractive one (resp. repelling when  $\eta < 0$ ), contracted (resp. expanded) by  $TP$  more sharply than  $TC_0$ .

Let now consider a perturbation of  $P$

$$Q(\theta, \rho) = (\theta + 2\pi\nu + \frac{1 - e^{-2\pi\eta}}{\eta}\rho + \varepsilon f(\theta, r), \rho e^{-2\pi\eta} + \varepsilon g(\theta, r)),$$

$f$  and  $g$  being bounded real analytic functions in their arguments.

We will show that the normal hyperbolicity of the invariant circle implies its persistence under perturbations of size  $\varepsilon$ , provided it is strong enough with respect to  $\varepsilon$ .

We start by remarking that there exists a positive constant  $C$  such that  $\rho' - \rho < 0$  outside the annulus  $|\rho| \leq C\varepsilon/\eta$ ; the normal behaviors of  $Q$  and  $P$  are hence analogue. It remains to refine the study inside of it and conclude the existence of an invariant circle.

**Theorem 2.1** (Existence of an invariant circle for  $Q$ ). *If  $\eta \gg \sqrt{\varepsilon}$ , the map  $Q$  possesses a unique invariant  $C^1$ -circle in the vicinity of  $C_0 = \mathbb{T} \times \{\rho = 0\}$ , whose basin of attraction coincides with  $\mathbb{T} \times \mathbb{R}$ .*

The proof, reposing on the following lemmata, concludes the section.

Since  $\rho = 0$  is the only invariant circle of  $P$ , we will find a unique invariant circle of  $Q$  as the fixed point of a graph transform on an opportune functional space; the dissipation makes the graph transform a contraction.<sup>6</sup>

We will consider  $\mathbb{T}$  with the usual induced topology. Let us take the compact  $\mathbb{T} \times [-1, 1]$  centered at  $\rho = 0$  and a Lipschitz map  $\varphi : \mathbb{T} \rightarrow [-1, 1]$ ,  $\theta \mapsto \varphi(\theta)$ , with  $\text{Lip } \varphi \leq k$ . We will call  $\text{Lip}_k$  the set of Lipschitz functions with Lipschitz constant less than or equal to  $k$ , and endow it with the  $C^0$ -metric.

Let  $\text{Gr } \varphi = \{(\theta, \varphi(\theta)) \in \mathbb{T} \times [-1, 1]\}$  be the graph of  $\varphi$ . For convenience, we have supposed that  $Q$  is defined everywhere, hence the composition  $Q(\text{Gr } \varphi)$  makes sense. We note  $Q(\theta, r) = (\Theta, R)$ . The components of  $Q(\theta, \varphi(\theta))$  are:

$$\begin{aligned} \Theta \circ (\text{id}, \varphi)(\theta) &= \theta + 2\pi\nu + \frac{1 - e^{-2\pi\eta}}{\eta}\varphi(\theta) + \varepsilon f(\theta, \varphi(\theta)) \\ R \circ (\text{id}, \varphi)(\theta) &= \varphi(\theta)e^{-2\pi\eta} + \varepsilon g(\theta, \varphi(\theta)). \end{aligned}$$

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<sup>6</sup>In this context we could take the term "normal hyperbolic" as the synonym of the contraction property of the graph transform associated to  $Q$ .

We define as usual the graph transform  $\Gamma : \varphi \mapsto \Gamma\varphi$  by

$$(2.3) \quad \Gamma\varphi : \theta \mapsto R \circ (\text{id}, \varphi) \circ [\Theta \circ (\text{id}, \varphi)]^{-1}(\theta).$$

The graph of  $\Gamma\varphi$  is the image by  $Q$  of the graph of  $\varphi$ :  $Q(\text{Gr } \varphi) = \text{Gr}(\Gamma\varphi)$  (see figure below); it is a classical tool for proving the existence of invariant attracting objects (see [17, 26, 30] at instance).

We look for a class of Lipschitz functions  $\text{Lip}_k$  such that  $\Gamma$  defines a contraction of  $\text{Lip}_k$  in the  $C^0$ -metric.

Although we are interested in small values of  $k > 0$  ( $\varepsilon$  being small, we do not expect the invariant curve to be in a class of functions with big variations) we will need  $k$  as well as  $\eta$  to be larger than  $\varepsilon$ . We will try to realize this for  $1 \gg \eta, k, \varepsilon$ , since if  $\eta$  is in the vicinity of 1, the persistence of the invariant circle is very easily shown.

**Remark 2.1.** *The graph transform method provides an invariant circle which a priori is no more than Lipschitz (in the sense that it will be the graph of a Lipschitz map of Lipschitz constant  $\leq k$ ). Actually more is true as the invariant circle is automatically at least  $C^1$ . We send the reader to [20] for a proof of this fact, based on the fiber contraction theorem. Moreover, when  $P$  is  $r$ -normally hyperbolic and  $Q$  is in a  $C^r$ -neighborhood of  $P$ , the invariant circle of  $Q$  is proved to be  $C^r$  (see again [20, §2-3] or [1, Corollary 2.2]).*

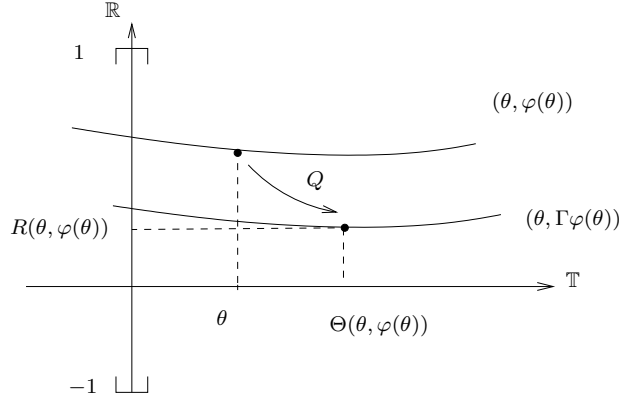


FIGURE 2. How the graph transform acts

Since  $f$  and  $g$  are real analytic on  $\mathbb{T} \times [-1, 1]$ , they are Lipschitz. First of all we have to guarantee the invertibility of  $\Theta \circ (\text{id}, \varphi) = \text{id} + u$ .

**Lemma 2.1.** *For every positive  $\eta$ , provided  $\varepsilon$  is sufficiently small,  $\Theta \circ (\text{id}, \varphi)$  is invertible.*



*Proof.* If  $u$  is a contraction,  $\text{id} + u$  is invertible with  $\text{Lip}(\text{id} + u)^{-1} \leq \frac{1}{1 - \text{Lip } u}$ . Using the definition of  $u$  and  $f$  being analytic, we have

$$\begin{aligned} |u(\theta_1) - u(\theta_2)| &\leq \text{Lip } \varphi \frac{(1 - e^{-2\pi\eta})}{\eta} |\theta_1 - \theta_2| + \varepsilon A_f |(\theta_1, \varphi(\theta_1)) - (\theta_2, \varphi(\theta_2))| \\ &\leq \left( \frac{(1 - e^{-2\pi\eta})}{\eta} k + \varepsilon A_f (1 + k) \right) |\theta_1 - \theta_2|, \end{aligned}$$

where  $A_f = \max \left( \sup_{\mathbb{T} \times [-1, 1]} |D_\theta f|, \sup_{\mathbb{T} \times [-1, 1]} |D_r f| \right)$ .

Since  $\varepsilon, k \ll 1$ ,  $\text{Lip } u < 1$ . □

**Lemma 2.2.** *The functions  $\Theta$  and  $R$  are Lipschitz on  $\mathbb{T} \times [-1, 1]$ .*

*Proof.* It is immediate from the expression of  $Q$  that for  $z_1, z_2$  in  $\mathbb{T} \times [-1, 1]$

$$|R(z_1) - R(z_2)| \leq (e^{-2\pi\eta} + \varepsilon A_g) |z_1 - z_2|$$

where  $A_g = \max \left( \sup_{\mathbb{T} \times [-1, 1]} |D_\theta g|, \sup_{\mathbb{T} \times [-1, 1]} |D_r g| \right)$ , and

$$|\Theta(z_1) - \Theta(z_2)| \leq \left( 1 + \frac{(1 - e^{-2\pi\eta})}{\eta} + \varepsilon A_f \right) |z_1 - z_2|.$$

□

**Lemma 2.3.** *The graph transform  $\Gamma$  is well defined from  $\text{Lip}_k$  to itself, provided  $k$  and  $\eta$  satisfy  $\varepsilon/\eta < k \ll \eta \ll 1$ .*

*Proof.* From the definition of the graph transform and the previous lemmata, we have

$$\begin{aligned} |\Gamma\varphi(\theta_1) - \Gamma\varphi(\theta_2)| &\leq \frac{\text{Lip } R \circ (\text{id}, \varphi)}{1 - \text{Lip } u} |\theta_1 - \theta_2| \\ &\leq \frac{ke^{-2\pi\eta} + \varepsilon A_g(1 + k)}{1 - \left[ \frac{1 - e^{-2\pi\eta}}{\eta} k + \varepsilon A_f(1 + k) \right]} |\theta_1 - \theta_2|. \end{aligned}$$

We want to find conditions on  $\eta$  and  $k$ , such that  $\varepsilon \ll 1$  being fixed,  $\Gamma$  is well defined in  $\text{Lip}_k$ ; we must satisfy

$$ke^{-2\pi\eta} + \varepsilon A_g(1 + k) \leq k \left\{ 1 - \left[ \frac{1 - e^{-2\pi\eta}}{\eta} k + \varepsilon A_f(1 + k) \right] \right\},$$

hence

$$k \left\{ 1 - e^{-2\pi\eta} - \left[ \frac{1 - e^{-2\pi\eta}}{\eta} k + \varepsilon A_f(1 + k) \right] \right\} \geq \varepsilon A_g(1 + k).$$

It suffices to choose  $k$  so that<sup>7</sup>

$$(2.4) \quad 1 \gg \eta \gg k \quad \text{with} \quad k > \frac{\varepsilon}{\eta}.$$

Clearly the larger  $\eta$  is, the easier it is to realize the inequality. □

The following technical lemma will be the key of the final proof.

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<sup>7</sup>Note that, despite one could expect  $\eta$  to be of order  $\varepsilon$ , in this case the  $O(1)$ -size of the twist  $\frac{1 - e^{-2\pi\eta}}{\eta}$  forces  $k$ , hence  $\eta$ , to be strong enough to satisfy the inequality.

**Lemma 2.4.** *Let  $z = (\theta, \rho)$  be a point in  $\mathbb{T} \times [-1, 1]$  and let  $\eta, k, \varepsilon$  satisfy condition (2.4). The following inequality holds for every  $\varphi \in \text{Lip}_k$ :*

$$|R(\theta, \rho) - \Gamma\varphi \circ \Theta(\theta, \rho)| \leq C|\rho - \varphi(\theta)|,$$

$C$  being a constant smaller than 1.

*Proof.* The following chain of inequalities holds:

$$\begin{aligned} |R(\theta, \rho) - \Gamma\varphi \circ \Theta(\theta, \rho)| &\leq |R(\theta, \rho) - R(\theta, \varphi(\theta))| + |R(\theta, \varphi(\theta)) - \Gamma\varphi \circ \Theta(\theta, \rho)| \\ &\leq \text{Lip } R |(\theta, \rho) - (\theta, \varphi(\theta))| + \text{Lip } \Gamma\varphi |\Theta(\theta, \varphi(\theta)) - \Theta(\theta, \rho)|, \end{aligned}$$

from the definition of  $\Gamma$ . We observe that

$$|\Theta(\theta, \varphi(\theta)) - \Theta(\theta, \rho)| \leq \left( \frac{1 - e^{-2\pi\eta}}{\eta} + \varepsilon A_f \right) |\varphi(\theta) - \rho| \leq (2\pi + \varepsilon A_f) |\varphi(\theta) - \rho|,$$

hence

$$|R(\theta, \rho) - \Gamma\varphi \circ \Theta(\theta, \rho)| \leq [\text{Lip } R + \text{Lip } \Gamma\varphi (2\pi + \varepsilon A_f)] |\varphi(\theta) - \rho|,$$

and this chain of inequalities holds

$$\begin{aligned} \text{Lip } R + \text{Lip } \Gamma\varphi (2\pi + \varepsilon A_f) &\leq \text{Lip } R + k(2\pi + \varepsilon A_f) \\ &\leq e^{-2\pi\eta} + \varepsilon A_g + k2\pi + \varepsilon k A_f \\ &= 1 - 2\pi\eta + O(\eta^2) + k2\pi + \varepsilon A_g + \varepsilon k A_f < 1 \end{aligned}$$

since (2.4) holds and, consequently,  $\eta \gg \varepsilon, k$  and  $\eta \gg k\varepsilon$ .  $\square$

*Proof of theorem 2.1.* We want to show that  $\Gamma$  defines a contraction in the space  $\text{Lip}_k$ : indeed  $\text{Lip}_k$  is a closed subspace of the Banach space  $C^0(\mathbb{T}, [-1, 1])$ , hence complete. The standard fixed point theorem then applies once we show that  $\Gamma$  is a contraction.

Let  $z$  be a point of  $\mathbb{T}$ , for every  $\varphi_1, \varphi_2$  in  $\text{Lip}_k$  we want to bound

$$|\Gamma\varphi_1(z) - \Gamma\varphi_2(z)|.$$

The trick is to introduce the following point in  $\mathbb{T} \times [-1, 1]$ ,

$$(\theta, \rho) = ([\Theta \circ (\text{id}, \varphi_1)]^{-1}(z), \varphi_1([\Theta \circ (\text{id}, \varphi_1)]^{-1}(z)))$$

and remark the following equality

$$\begin{aligned} \Gamma\varphi_2 \circ \Theta(\theta, \rho) &= \Gamma\varphi_2(\Theta([\Theta \circ (\text{id}, \varphi_1)]^{-1}(z), \varphi_1([\Theta \circ (\text{id}, \varphi_1)]^{-1}(z)))) \\ &= R \circ (\text{id}, \varphi_2) \circ [\Theta \circ (\text{id}, \varphi_2)]^{-1} \circ [\Theta \circ (\text{id}, \varphi_1)] [\Theta \circ (\text{id}, \varphi_1)]^{-1}(z) \\ &= R \circ (\text{id}, \varphi_2) \circ [\Theta \circ (\text{id}, \varphi_2)]^{-1}(z) = \Gamma\varphi_2(z). \end{aligned}$$

We hence apply lemma 2.4 to  $\varphi = \varphi_1$  at the point  $(\theta, \rho)$  previously introduced. We have

$$|\Gamma\varphi_1(z) - \Gamma\varphi_2(z)| \leq C |\varphi_1 \circ [\Theta \circ (\text{id}, \varphi_1)]^{-1}(z) - \varphi_2 \circ [\Theta \circ (\text{id}, \varphi_1)]^{-1}(z)|.$$

Taking the supremum for all  $z$  and remembering that  $C < 1$ , concludes the proof of the theorem.  $\square$

## 3. SECOND LOCALIZATION

We now consider the part of the  $(\eta, \nu)$ -plane defined by  $|\eta| \ll 1$  in which the graph transform does not work a priori and perform convenient coordinate's changes on  $Q$  in order to write it in a normal form that will allow to conclude again the existence of an invariant circle.

**3.1. Translated circle of rotation  $2\pi\alpha$ .** In section 2.1 we have localized our study at the circle  $\rho = 0$ ; we now want to focus on the translated one with a given Diophantine rotation  $2\pi\alpha$ .

To do so, note that the translation function  $\tau = 2\pi\eta(\nu - \alpha)$  defines a family of hyperbolas in the  $(\eta, \nu)$ -plane. In the terms of  $(\tau, \eta)$ ,  $P$  becomes

$$(3.1) \quad P(\theta, \rho) = \left( \theta + 2\pi\alpha + \frac{\tau}{\eta} + \frac{1 - e^{-2\pi\eta}}{\eta} \rho, \rho e^{-2\pi\eta} \right);$$

performing the change of variables

$$(\theta, \rho) \mapsto \left( \theta, \rho - \frac{2\pi\eta(\nu - \alpha)}{e^{-2\pi\eta} - 1} = \tilde{\rho} \right),$$

we get

$$P(\theta, \tilde{\rho}) = \left( \theta + 2\pi\alpha + \frac{1 - e^{-2\pi\eta}}{\eta} \tilde{\rho}, \tilde{\rho} e^{-2\pi\eta} + \tau \right).$$

Considering the corresponding perturbed diffeomorphism

$$Q(\theta, \tilde{\rho}) = \left( \theta + 2\pi\alpha + \frac{1 - e^{-2\pi\eta}}{\eta} \tilde{\rho} + \varepsilon f(\theta, \tilde{\rho}), \tilde{\rho} e^{-2\pi\eta} + \tau + \varepsilon g(\theta, \tilde{\rho}) \right),$$

we want prove that there exists an invariant circle even for values of  $\eta$  smaller than the ones given in theorem 2.1 ( $1 \gg \eta \gg \sqrt{\varepsilon}$ ).

Let us first remark that there exists a positive constant  $\tilde{C}$  such that outside the annulus containing the translated circle  $\tilde{\rho} = 0$  of  $P$

$$A^- = \left\{ (\theta, r), |\tilde{\rho}| \leq \frac{\tau}{1 - e^{-2\pi\eta}} + \tilde{C}\varepsilon/\eta \right\}$$

we have  $\tilde{\rho}' - \tilde{\rho} < 0$ .

In order to conclude the existence of an invariant circle inside of it, we first localize at the translated circle of  $Q$  close to  $\tilde{\rho} = 0$  (whose existence is guaranteed by Rüssmann's theorem), then write  $Q$  in a more refined form that allows to take advantage of the normal attraction even inside  $A^-$  and apply the graph transform again. Let us proceed with this scheme.

Rüssmann's theorem B.1 guarantees, if  $2\pi\alpha$  is Diophantine, if  $Q$  has torsion (in this case  $\frac{1 - e^{-2\pi\eta}}{\eta} \rightarrow 2\pi$ , when  $\eta \rightarrow 0$ ) and is sufficiently close to  $P$ , the existence of an analytic curve  $\gamma : \mathbb{T} \rightarrow \mathbb{R}$ , a diffeomorphism of the torus  $h$  and  $\lambda \in \mathbb{R}$  such that

- the image of the curve  $\tilde{\rho} = \gamma(\theta)$  via  $Q$ , is the translated curve of equation  $\tilde{\rho} = \lambda + \gamma(\theta)$
- the restriction of  $Q$  to  $\text{Gr } \gamma$  is conjugated to the rotation  $R_{2\pi\alpha} : \theta \mapsto \theta + 2\pi\alpha$ .

For a statement of this result in  $C^\infty$  topology see [2, 32] for example, in finite differentiability see the original paper of Rüssmann [27] or the works of Herman [18], where a generalization of the theorem to the case of rotation numbers of constant type is also proved. Since here we deal with real analytic diffeomorphisms, we will use the theorem in its analytic version, given in [24, Appendix A] as the consequence of a discrete time analogue of Moser's normal form theorem [25]. Rüssmann's statement is recalled in Appendix B where, as a byproduct, we show that if  $\eta$  is larger than  $c_0\varepsilon$ ,  $c_0$  being a positive constant, it is still possible to eliminate the translation  $\lambda$ , along some curve of parameters, even for generic perturbations  $Q$  of  $P$ .

In the conditions of applicability of Rüssmann's theorem, the local diffeomorphism

$$G : (\theta, \tilde{\rho}) \mapsto (h^{-1}(\theta) = \xi, \tilde{\rho} - \gamma(\theta) = x),$$

sends  $\tilde{\rho} = \gamma(\theta)$  to  $x = 0$  and is such that  $G \circ Q \circ G^{-1}$  has  $x = 0$  as a translated curve on which the dynamics is the rotation of angle  $2\pi\alpha$ . We have:

$$\begin{cases} Q(\xi, x) = (\xi', x') \\ \xi' = h^{-1}\left(h(\xi + 2\pi\alpha) + \frac{1-e^{-2\pi\eta}}{\eta}x + \varepsilon \sum_{j=1} \frac{1}{j!} \frac{\partial^j f}{\partial \tilde{\rho}^j}(\theta, \gamma(\theta)) x^j\right), \\ x' = \lambda + xe^{-2\pi\eta} + \gamma(h(\xi + 2\pi\alpha)) - \gamma(h(\xi + 2\pi\alpha) + \frac{1-e^{-2\pi\eta}}{\eta}x + O(|x|)) + \\ + \varepsilon \sum_{i=1} \frac{1}{i!} \frac{\partial^i g}{\partial \tilde{\rho}^i}(\theta, \gamma(\theta)) x^i, \end{cases}$$

hence

$$(3.2) \quad Q(\xi, x) = (\xi + 2\pi\alpha + \sum_i A_i(\xi) x^i, \lambda + \sum_i B_i(\xi) x^i),$$

where

- $B_1(\theta) = e^{-2\pi\eta} - D\gamma(h(\xi + 2\pi\alpha)) \cdot \left(\frac{1-e^{-2\pi\eta}}{\eta} + \varepsilon \frac{\partial f}{\partial \tilde{\rho}}(\theta, \gamma(\theta))\right) + \varepsilon \frac{\partial g}{\partial \tilde{\rho}}(\theta, \gamma(\theta))$ , hence it is of order  $1 + O(\varepsilon)$ ,
- $B_i(\theta)$ , for  $i > 1$ , is the coefficient of the order- $i$  term in  $x$  from the development of terms as

$$-\frac{1}{i!} D^i \gamma(h(\xi + 2\pi\alpha)) \cdot \left(\frac{1-e^{-2\pi\eta}}{\eta}x + \varepsilon \sum_{j=1} \frac{1}{j!} \frac{\partial^j f}{\partial \tilde{\rho}^j}(\theta, \gamma(\theta)) x^j\right)^i + \varepsilon \frac{1}{i!} \frac{\partial^i g}{\partial \tilde{\rho}^i}(\theta, \gamma(\theta)),$$

and has order  $O(\varepsilon)$ .

- $A_i(\theta)$  is the order- $i$  term coming from

$$\frac{1}{i!} D^i h^{-1}(h(\xi + 2\pi\alpha)) \cdot \left(\frac{1-e^{-2\pi\eta}}{\eta}x + \varepsilon \sum_{j=1} \frac{1}{j!} \frac{\partial^j f}{\partial \tilde{\rho}^j}(\theta, \gamma(\theta)) x^j\right)^i.$$

In particular  $A_i(\theta)$  is of order  $1 + O(\varepsilon)$  for  $i = 1$  and  $O(\varepsilon)$  otherwise.

We noted  $\theta = h(\xi)$  and omitted indices indicating the smooth dependence on  $\alpha, \eta$  and  $\tau$ .

This change of coordinates actually permits us to see  $Q$  as the composition of a map

$$\mathcal{I}_{\eta, \tau} = (\xi + 2\pi\alpha + \sum_i A_i(\xi) x^i, \sum_i B_i(\xi) x^i),$$

leaving the circle  $x = 0$  invariant, with a translation  $T_\lambda : (\theta, r) \mapsto (\theta, \lambda + r)$  in the normal direction. Remark that when  $\varepsilon = 0$ , we have  $h = \text{id}$ ,  $\gamma = 0$  and  $\lambda = \tau$ , thus  $Q$  would read as before the perturbation; in addition even if we don't dispose of the explicit form of the translation function  $\lambda$ , the implicit function theorem tells us that  $\lambda = \tau + O(\varepsilon)$ .

**3.2. Normally attractive invariant circles, again.** We now write  $Q$  in a form that entails the existence of an invariant circle and delimit regions in the space of parameters in which the normal hyperbolicity is still strong enough to guarantee its persistence.

If  $\lambda \neq 0$ , it seems impossible to write  $Q$  in a form as gentle as  $\mathcal{I}_{\eta, \tau}$ . The idea is to use all the strength of the translation  $\lambda$ : we perform coordinates changes that push the dependence on the angles as far as possible, let say up to a certain order  $k$ , and eventually remark that **all the dependence on the angles of the remaining terms will cancel out with  $\lambda$** .

**Theorem 3.1.** *If  $\alpha$  is Diophantine, there exists a real analytic change of variables that transforms  $Q$  into the normal form:*

$$(3.3) \quad \begin{cases} Q(\Theta, R) = (\Theta', R') \\ \Theta' = \Theta + 2\pi\alpha + \sum_{i=1}^k \bar{\alpha}_i R^i + O(\varepsilon|R|^{k+1}) + O(|\lambda|\varepsilon) \\ R' = \lambda + \bar{\beta}_1 R + \sum_{i=2}^k \bar{\beta}_i R^i + O(\varepsilon|R|^{k+1}) + O(|\lambda|\varepsilon), \end{cases}$$

$\bar{\alpha}_i$  and  $\bar{\beta}_i$  being constants,  $O(\varepsilon|R|^{k+1})$  being terms of order  $\geq k+1$  vanishing when  $R = 0$ ,  $O(|\lambda|\varepsilon)$  being terms of order  $O(\tau\varepsilon) + O(\varepsilon^2)$ , possibly depending on  $\Theta$  but vanishing when  $\lambda = 0$ .

In particular, for values of the parameters belonging to the regions defined by  $\eta \gg \varepsilon$  and  $|\tau| \leq \eta^2$ ,  $Q$  possesses an invariant normally-attractive (resp. repulsive if  $\eta < 0$ )  $C^1$ -circle.

The proof is divided in two steps. A note discussing further regularity of the invariant circle is found at the end, as complement to the proof.

*Step 1: normal form.* In order to write  $Q$  in a normal form like (3.3), we do an extensive use of the Diophantine property of  $\alpha$  (1.3) and repeatedly apply lemma A.1.

Using the fact that  $B_1(\xi)$  is close to 1, we see that the difference equation

$$(3.4) \quad \log B_1(\xi) + \log X(\xi) - \log X(\xi + 2\pi\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \log B_1(\xi) d\xi$$

then has a unique analytic solution  $X(\xi)$  close to 1.

Hence, the coordinates change

$$(\xi, x) \mapsto (\xi, \frac{x}{X(\xi)} = y)$$

transforms  $Q$  into a map of the form

$$\begin{cases} Q(\xi, y) = (\xi', y') \\ \xi' = \xi + 2\pi\alpha + \sum_{i=1}^k \alpha_i(\xi) y^i + O(\varepsilon|y|^{k+1}) \\ y' = \lambda + \bar{\beta}_1 y + \sum_{i=2}^k \beta_i(\xi) y^i + O(\varepsilon|y|^{k+1}) + O(\varepsilon|\lambda||y|) + O(\varepsilon|\lambda|), \end{cases}$$

where

$$\bar{\beta}_1 = \exp \frac{1}{2\pi} \int_0^{2\pi} \log B_1(\xi) d\xi = 1 - 2\pi\eta + 2\pi^2\eta^2 + \varepsilon M_1 + \varepsilon^2 M_2 + O(\varepsilon\eta) + O(\varepsilon^3, \eta^3),$$

$M_i$  being constants coming from the average of the order- $\varepsilon^i$  terms in the Taylor's expansion of  $\log B_1(\xi)$ .

Just as for (3.4), there is a unique analytic solution  $X^{(2)}(\xi)$ , smoothly depending on the parameters - through  $\bar{\beta}_1$  -, of the equation

$$\bar{\beta}_1^2 X^{(2)}(\xi + 2\pi\alpha) - \bar{\beta}_1 X^{(2)}(\xi) + \beta_2(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \beta_2(\xi) d\xi = \bar{\beta}_2.$$

The change of variables

$$(\xi, y) \mapsto (\xi, y + X^{(2)}(\xi) y^2)$$

then transforms the non constant coefficient  $\beta_2(\xi)$  into its average  $\bar{\beta}_2$ .

Generalizing, by composing the following changes of variables

$$\begin{cases} (\xi, y) \mapsto (\xi, y + X^{(i)}(\xi) y^i) & i = 2, \dots, k \\ \bar{\beta}_1^i X^{(i)}(\xi + 2\pi\alpha) - \bar{\beta}_1 X^{(i)}(\xi) + \beta_i(\xi) = \bar{\beta}_i \end{cases}$$

and

$$\begin{cases} (\xi, y) \mapsto (\xi + Z^{(i)}(\xi) y^i, y) & i = 1, \dots, k \\ \bar{\beta}_1^i Z^{(i)}(\xi + 2\pi\alpha) - Z^{(i)}(\xi) + \alpha_i(\xi) = \bar{\alpha}_i \end{cases}$$

we are able to put  $Q$  in the form entailed in (3.3)

$$\begin{cases} Q(\Theta, R) = (\Theta', R') \\ \Theta' = \Theta + 2\pi\alpha + \sum_{i=1}^k \bar{\alpha}_i R^i + O(\varepsilon|R|^{k+1}) + O(|\lambda|\varepsilon) \\ R' = \lambda + \bar{\beta}_1 R + \sum_{i=2}^k \bar{\beta}_i R^i + O(\varepsilon|R|^{k+1}) + O(|\lambda|\varepsilon), \end{cases}$$

where  $\bar{\alpha}_1$  and  $\bar{\beta}_1$  are of order  $1 + O(\varepsilon)$  while  $\bar{\alpha}_i, \bar{\beta}_i$  for  $i > 1$ , of order  $O(\varepsilon)$ .

We thus have been able to confine the angle's dependency entirely in the terms  $O(\dots)$ ; in particular the terms  $O(|\lambda|\varepsilon)$  vanish when no translation occurs.

*Step 2: region of normal hyperbolicity.* To see that whenever  $\eta \gg \varepsilon$  and  $|\tau| \leq \eta^2$ , the diffeomorphism  $Q$  possesses an invariant normally-attractive (resp. repulsive if  $\eta < 0$ ) circle, it just remains to harvest the consequences of the normal form (3.3). This is an improvement (with respect to the previous result, valid for all kind of frequency) in terms of the minimal admissible size of  $\eta$  ensuring the persistence of an invariant circle by normal hyperbolicity.

The diffeomorphism  $Q$  is a perturbation of the normal form

$$N_{\eta, \tau}(\Theta, R) = (\Theta + 2\pi\alpha + \sum_{i=1}^k \bar{\alpha}_i R^i, \lambda(\tau, \varepsilon) + \sum_{i=1}^k \bar{\beta}_i R^i),$$

which possesses an invariant circle  $R = R_0$ , solution of  $R = \lambda + \sum_{i=1}^k \bar{\beta}_i R^i$ .

Using the implicit function theorem and the structure of the terms  $\bar{\beta}_1$  and  $\bar{\beta}_2$ , we have

$$(3.5) \quad R_0 = \frac{-\lambda}{\bar{\beta}_1 - 1} + O\left(\frac{|\lambda|^2 |\bar{\beta}_2|}{|\bar{\beta}_1 - 1|^3}\right) = R_- + O(\varepsilon \frac{|R_-|^2}{|\bar{\beta}_1 - 1|}),$$

where  $R_-$  reads more explicitly as

$$R_- = \frac{-\tau + O(\varepsilon)}{-2\pi\eta + \varepsilon M + O(\varepsilon\eta) + O(\eta^2, \varepsilon^2)}.$$

In order to see that it is still possible to apply the graph transform method and prove the existence of a normally hyperbolic invariant circle close to  $R_0$ , we perform a last change of variables:

$$(\Theta, R) \mapsto (\Theta, R - R_0 = \tilde{R}).$$

Now centered at  $R_0$ , the diffeomorphism  $Q$  reads

$$\begin{cases} Q(\Theta, \tilde{R}) = (\Theta', \tilde{R}') \\ \Theta' = \Theta + 2\pi\alpha + \bar{\alpha}_1 R_0 + \sum_{i=1}^k \bar{\alpha}_i \tilde{R}^i + O(\varepsilon|R_0||\tilde{R}|) + O(\varepsilon|\tilde{R}|^{k+1}) + O(\varepsilon|R_0|^2) + O(\varepsilon|\lambda|) \\ \tilde{R}' = (\bar{\beta}_1 + \sum_{i=2}^k i \bar{\beta}_i R_0^{i-1}) \tilde{R} + O(\varepsilon|R_0||\tilde{R}|^2) + O(\varepsilon|\tilde{R}|^2) + O(\varepsilon|R_0|^2) + O(\varepsilon|\lambda|). \end{cases}$$

Now  $\tilde{R} = 0$  is the invariant circle of the normal form, and the terms  $O(\varepsilon|R_0|^2) + O(\varepsilon|\lambda|)$  represent perturbations.

To better see, let us write explicitly the order 1 term in  $\tilde{R}$ :

$$(3.6) \quad \tilde{R}' = (1 - 2\pi\eta + \varepsilon M_1 + O(\varepsilon\eta) + \sum_{i=2}^k i \bar{\beta}_i R_0^{i-1}) \tilde{R} + O(\dots)$$

In the region defined by

$$(3.7) \quad \begin{cases} \eta \geq \sqrt{2\pi}|\nu - \alpha|, & \text{hence } |\tau| \leq \eta^2 \\ \eta \gg \varepsilon, \end{cases}$$

the term  $\bar{\beta}_2 R_0$  is of order  $O(\varepsilon\eta) + \frac{O(\varepsilon^2)}{O(\eta)} \ll \varepsilon$ .

We remark in particular that each region of this type actually contains the curve  $C_\alpha$  along which  $\nu = \alpha + O(\varepsilon^2)$ .

**Remark 3.1.** *The regions we defined above enlarge the known domain of normal hyperbolicity which, up to now, we know to include values of  $\eta \gg \sqrt{\varepsilon}$ . Nevertheless, there is little hope to draw these regions till  $\eta = 0$ , the terms  $\varepsilon^i M_i$  constituting an obstruction to the normal hyperbolicity, which would be guaranteed if  $(1 - 2\pi\eta)$  dominated over the rest.*

Moreover, not even the first order term of the time- $\varepsilon$  flow of the spin-orbit system

$$v = \begin{cases} \dot{\theta} = \alpha + r \\ \dot{r} = -\eta r + \eta(\nu - \alpha) - \varepsilon f_\theta(\theta, t), \end{cases}$$

hints anything about the nullity of, at least, the first term  $\varepsilon M$ , once we impose the only exploitable information we have: the corresponding flow is conformally symplectic, as the divergence of  $v$  is equal to the constant  $-\eta$ .

In addition, even supposing that  $Q$  lives in the class of those flows for which  $\varepsilon^i M_i = 0$ ,  $\eta$  won't be allowed to reach 0; the first term would be

$$\tilde{R}' = (1 - 2\pi\eta + \frac{O(\varepsilon^2)}{O(\eta)} + O(\varepsilon\eta)) \tilde{R} + \dots,$$

and if we want  $1 - 2\pi\eta$  to dominate,  $\eta$  still has to satisfy  $\eta \gg \varepsilon$ . Our regions would then stop at a certain point and cannot follow tightly the  $C'_\alpha$ s till the end.

In the region defined by  $|\tau| \leq \eta^2$  and  $\eta \gg \varepsilon$ ,  $R_0$  is of order  $O(\eta) + O(\varepsilon/\eta)$  and  $O(\lambda) = O(\eta^2) + O(\varepsilon)$ ; more specifically,

$$\begin{cases} Q(\Theta, \tilde{R}) = (\Theta', \tilde{R}') \\ \Theta' = \Theta + 2\pi\alpha + \bar{\alpha}_1 R_0 + (C + O(\varepsilon))\tilde{R} + O(\varepsilon|\tilde{R}|^2) + O(\varepsilon|R_0|^2) + O(\varepsilon^2) + O(\varepsilon\eta^2) \\ \tilde{R}' = (1 - 2\pi\eta + O(\varepsilon))\tilde{R} + O(\varepsilon|\tilde{R}|^2) + O(\varepsilon|R_0|^2) + O(\varepsilon^2) + O(\varepsilon\eta^2), \end{cases}$$

having denoted by  $C$  the twist  $\frac{1-e^{-2\pi\eta}}{\eta}$ .

It is evident that  $Q = \tilde{N} + P^\varepsilon$ , is a perturbation of

$$\begin{cases} \tilde{N}(\Theta, \tilde{R}) = (\Theta', \tilde{R}') \\ \Theta' = \Theta + 2\pi\alpha + \bar{\alpha}_1 R_0 + (C + O(\varepsilon))\tilde{R} + O(\varepsilon|\tilde{R}|^2) \\ \tilde{R}' = (1 - 2\pi\eta + O(\varepsilon))\tilde{R} + O(\varepsilon|\tilde{R}|^2). \end{cases}$$

In particular the term  $O(\varepsilon|R_0|^2)$  is constant and the perturbation  $P^\varepsilon$  satisfies

$$P^\varepsilon = O(\varepsilon|R_0|^2) + O(\varepsilon^2) + O(\varepsilon\eta^2) \ll \varepsilon \ll \eta,$$

in the considered region.

To apply the graph transform method in the annulus  $|\tilde{R}| \leq 1$ , containing the circle  $R = R_0$ , is now an easy matter. The preponderance of  $1 - 2\pi\eta$  with respect to the reminder terms in the regions considered makes the procedure work and guarantee the existence of a normally-attractive (resp. repulsive) circle in a neighborhood of  $R_0$ .  $\square$

Note. The perturbation terms  $O(\dots)$  are valid in every  $C^r$ -topology (constants would depend on  $r$  though); moreover the invariant circle  $C_0 = \mathbb{T} \times \{0\}$  of  $\tilde{N}$  is actually  $r$ -normally hyperbolic: an immediate calculation shows that condition (2.2) is easily satisfied, since in the considered region for all  $x \in C_0$

$$\sup_x \|T\tilde{N}|_{N_x^s}\| = 1 - 2\pi\eta + O(\varepsilon) < \inf_x m(T\tilde{N}|_{T_x C_0})^r = (1 + O(\varepsilon))^r.$$

The invariant circle given by the graph transform is then  $C^r$ , but  $\varepsilon$  has to be intended all the smaller as  $r$  grows.

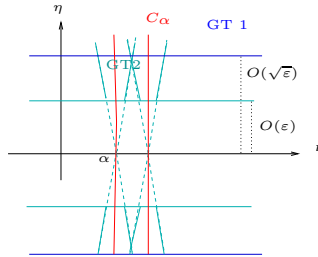


FIGURE 3. Graph transform improvement



When  $\eta \gg \varepsilon, |\tau| \leq \eta^2$  the successive localizations allowed to control the attractive dynamics of  $Q$  once entered  $A^-$ , thus providing global control on the basin of attraction of the invariant circle where the dynamics finally stuck at.

Besides our regions are defined up to an  $O(\varepsilon)$ -strip in the plane, by stability of the normal hyperbolicity, we know a priori that in a thin cusp region along every  $C_\alpha$  we can guarantee the persistence of invariant attractive/repelling curve. Unfortunately, our knowledge of  $C_\alpha$  is not explicit enough to allow a quantitative description of this thin neighborhood. Alternative topological arguments such as Morse index theory or the Wazevsky theorem would still provide answers for values of  $\eta$  up to order  $\varepsilon$ , thus preventing us to say which region contains the other in the  $O(\varepsilon)$ -strip.

For generic perturbations the dynamics contained in this strip is expected to be very rich: in a further study the existence of Birkhoff attractors and Aubry-Mather sets is likely to be proven.

#### APPENDIX A. DIFFERENCE EQUATION ON THE TORUS

Consider the complex extension  $\mathbb{T}_\mathbb{C} = \mathbb{C}/2\pi\mathbb{Z}$  of the torus  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$  and the corresponding  $s$ -neighborhood defined using  $\ell^\infty$ -balls (in the real normal bundle of the torus):

$$\mathbb{T}_s = \{\theta \in \mathbb{T}_\mathbb{C} : |\operatorname{Im} \theta| \leq s\}$$

Let  $\mathcal{A}(\mathbb{T}_s)$  be the space of holomorphic functions  $f : \mathbb{T}_s \rightarrow \mathbb{C}$ , endowed with the Banach norm

$$|f|_s = \sup_{\mathbb{T}_s} |f(\theta)|.$$

**Lemma A.1.** *Let  $\alpha$  be Diophantine in the sense of (1.3),  $g \in \mathcal{A}(\mathbb{T}_{s+\sigma})$  and let some constants  $a, b \in \mathbb{R} \setminus \{0\}$  be given. There exist a unique  $f \in \mathcal{A}(\mathbb{T}_s)$  of zero average and a unique  $\mu \in \mathbb{R}$  such that the following is satisfied*

$$(A.1) \quad \mu + af(\theta + 2\pi\alpha) - bf(\theta) = g(\theta), \quad \mu = \frac{1}{2\pi} \int_{\mathbb{T}} g(\theta) d\theta.$$

In particular  $f$  satisfies

$$|f|_s \leq \frac{C}{\gamma\sigma^{\tau+1}} |g|_{s+\sigma},$$

$C$  being a constant depending on  $\tau$ .

*Proof.* Developing in Fourier series one has

$$\mu + \sum_k (a e^{i2\pi k \alpha} - b) f_k e^{ik\theta} = \sum_k g_k e^{ik\theta},$$

we get  $\mu = g_0 = \frac{1}{2\pi} \int_{\mathbb{T}} g(\theta) d\theta$  and

$$f(\theta) = \sum_{k \neq 0} \frac{g_k}{a e^{i2\pi k \alpha} - b} e^{ik\theta}.$$

Remark that

$$\begin{aligned} |a e^{i2\pi k\alpha} - b|^2 &= (a-b)^2 \cos^2 \frac{2\pi k\alpha}{2} + (a+b)^2 \sin^2 \frac{2\pi k\alpha}{2} \\ &\geq (a+b)^2 \sin^2 \frac{2\pi k\alpha}{2} = (a+b)^2 \sin^2 \frac{2\pi(k\alpha - l)}{2}, \end{aligned}$$

with  $l \in \mathbb{Z}$ . Choosing  $l \in \mathbb{Z}$  such that  $\frac{2\pi(k\alpha - l)}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ , we get

$$|a e^{i2\pi k\alpha} - b| \geq \frac{\pi^2}{4} |a+b| |k\alpha - l| \geq \frac{\pi^2}{4} |a+b| \frac{\gamma}{|k|^\tau},$$

using that  $|\sin x| \geq \frac{\pi}{2}|x|$ ,  $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  and condition (1.3). Hence the lemma.  $\square$

We address the reader interested to optimal estimates to [28].

#### APPENDIX B. THE TRANSLATED CURVE OF RÜSSMANN

The proof of this statement in analytic category - used in the present paper - is given in [24, Appendix A].

We consider real analytic diffeomorphisms in  $\mathbb{T} \times \mathbb{R}$  that, in the neighborhood of the circle  $T_0 = \mathbb{T} \times \{r = 0\}$ , can be expressed as

$$Q(\theta, r) = (\theta + 2\pi\alpha + t(r) + f(\theta, r), (1 + A) \cdot r + g(\theta, r));$$

where  $t(0) = 0$  and  $t'(r) > 0$  for every  $r$ ,  $\alpha \in \mathbb{R}$  satisfies the Diophantine condition for some  $\gamma, \tau > 0$

$$|k\alpha - l| \geq \frac{\gamma}{|k|^\tau} \quad \forall (k, l) \in \mathbb{N} \setminus \{0\} \times \mathbb{Z},$$

and  $A$  is a positive or negative real constant. If  $A \neq 0$ ,  $T_0$  is a normally hyperbolic invariant circle of rotation  $2\pi\alpha$  of

$$(B.1) \quad P^0(\theta, r) = (\theta + 2\pi\alpha + O(r), (1 + A)r + O(r^2)),$$

of which  $Q$  represents a perturbation.

**Theorem B.1** (Rüssmann). *Let  $\alpha$  be Diophantine and*

$$P^0(\theta, r) = (\theta + 2\pi\alpha + t(r) + O(r^2), (1 + A)r + O(r^2))$$

*such that  $t(0) = 0$  and  $t'(r) > 0$ .*

*If  $Q$  is close enough to  $P^0$  there exists a unique real analytic curve  $\gamma \in \mathcal{A}(\mathbb{T}, \mathbb{R})$ , close to  $r = 0$ , a diffeomorphism  $h$  of  $\mathbb{T}$  close to the identity and fixing the origin, and  $b \in \mathbb{R}$ , close to 0, such that*

$$Q(\theta, \gamma(\theta)) = (h \circ R_{2\pi\alpha} \circ h^{-1}(\theta), b + \gamma(h \circ R_{2\pi\alpha} \circ h^{-1}(\theta))).$$

Actually in its original version the theorem is stated for  $A = 0$ ; to consider the more general case with  $A$  close to 0, does not bring any further difficulties. The proof is based on a suitable inverse function theorem in analytic category.

**B.1. Curves  $C_\alpha$  for general perturbations of the unperturbed spin-orbit time  $2\pi$ -flow.** With no further assumptions on  $Q$ , one cannot a priori expect that the translation  $b$  vanishes in some circumstances. If in the case of vector fields relative to the spin-orbit problem, the Hamiltonian structure of equations and the dependence on the external parameter  $\nu \in \mathbb{R}$  has been the key point to solve  $b = 0$  and obtain the dynamical conjugacy (see [23, Sections 6.2 and 7]), in this study we consider generic analytic perturbation of the time  $2\pi$ -map relative to the unperturbed spin-orbit equations:

$$Q(\theta, \tilde{\rho}) = (\theta + 2\pi\alpha + \frac{1 - e^{-2\pi\eta}}{\eta} \tilde{\rho} + \varepsilon f(\theta, \tilde{\rho}), e^{-2\pi\eta} \tilde{\rho} + \tau + \varepsilon g(\theta, \tilde{\rho})).$$

In this case, nothing ensures that the  $C_\alpha$  portrayed in the very particular context of the vector-field of the spin-orbit, exist and reach the  $\eta = 0$  axis (the hamiltonian axis). However, if these curves are not expected to exist for any value of  $\eta$  and  $\varepsilon$ , it is easy to prove their existence provided  $\eta$  is stronger than the perturbation. As a matter of fact, we notice that when no perturbation occurs  $Q$  reduces to

$$P(\theta, \tilde{\rho}) = (\theta + 2\pi\alpha + \frac{1 - e^{-2\pi\eta}}{\eta} \tilde{\rho}, e^{-2\pi\eta} \tilde{\rho} + \tau),$$

and the circle  $\tilde{\rho} = 0$  undergoes the translation  $b := \tau = 2\pi\eta(\nu - \alpha)$ . For the unique choice of the parameter  $\nu = \alpha$ ,  $\mathbb{T} \times \{0\}$  is invariant.

For any  $\eta \in [-\eta_0, \eta_0]$ ,  $\eta_0 \in \mathbb{R}^+$ , Rüssman's theorem applied to  $Q$ , sufficiently close to  $P$ , provides  $b, \gamma$  and  $h$  (analytic in  $\varepsilon$  and smooth in  $\eta, \nu$ ) such that

$$Q(\theta, \gamma(\theta)) = (h \circ R_{2\pi\alpha} \circ h^{-1}(\theta), b + \gamma(h \circ R_{2\pi\alpha} \circ h^{-1}(\theta))).$$

In order to prove that  $b = 0$  implicitly defines  $\nu$ , it suffices to show that  $\nu \mapsto b(\nu)$  is a local diffeomorphism; since this is an open property with respect to the  $C^1$ -topology, and  $Q$  is close to  $P$ , it suffices to show it for  $P$ , which is immediate. Note that  $b = \tau + O(\varepsilon) = 2\pi\eta(\nu - \alpha) + O(\varepsilon)$ , and because of the regularity of  $b$ , there is a positive constant  $M$ , independent of  $\eta$  and  $\nu$ , such that whenever  $|\eta| > M\varepsilon$ ,  $|b'|$  is bounded away from 0. By the implicit function theorem in finite dimension, there exists a unique  $\nu$  close to  $\alpha$ , such that  $b = 0$ .

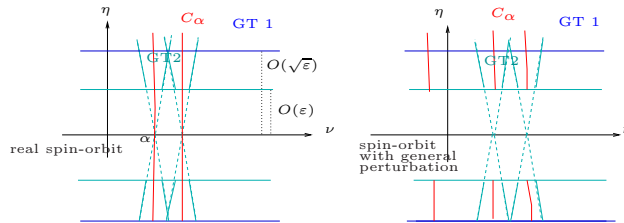


FIGURE 4. Two situations: for the real spin-orbit flow curves reach the Hamiltonian axis, they stop at the order  $O(\varepsilon)$  otherwise.

**Acknowledgments.** *This work would have never seen the light without the mathematical (and moral) support and advises of A. Chenciner and J. Féjoz all the way through my Ph.D. I'm grateful and in debt with both of them. Thank you to T. Castan, B. Fayad, J. Laskar, J.-P. Marco, P. Mastrolia, L. Niederman and P. Robutel for the enlightening discussions we had and their constant interest in this (and future) work.*

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